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The Spectrum of an Invariant Submanifold*

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This paper is concerned with vector fields on smooth compact manifolds. The exponential growth of solutions of the linearized equations is described by the already well-known Spectral Theorem applied to the induced linear flow on the tangent bundle. The spectrum of the tangent bundle flow is compared to the two secondary spectra obtained by first taking the spectrum of the bundle of tangent spaces to an invariant submanifold and second, the spectrum of an induced flow on an arbitrary complementary bundle to the latter. The relationship among the three spectra is studied and it is shown that whenever these secondary spectra are disjoint then an invariant complementary bundle can be found. The results have implications in the theory of perturbation of invariant manifolds. The problem is studied in the setting of skew-product dynamical systems and the results are applicable to block triangular systems of ordinary differential equations.

I. INTRODUCTION

The primary objective in this paper is to introduce the concept of the spectrum of a compact submanifold Y in a Riemannian manifold M, where Y is an invariant set in the flow σ generated by a C^1-vector field V on M. Roughly speaking the spectrum describes the exponential growth rates of solutions of the vector field V in the vicinity of Y. The theory we shall describe here is based upon the notion of the spectrum of a linear skew-product flow π on a vector bundle $\mathscr{E}$, a theory which was developed in [26]. In this setting the vector bundle $\mathscr{E}$ is the tangent bundle TM restricted to the base space Y, and the flow π is the linearized flow induced on TM by the vector field V. The bundle $\mathscr{E}$ is an invariant set for π. Also the tangent bundle $\mathscr{H} = TY$ is another invariant set, and $\mathscr{H}$ is a subbundle of $\mathscr{E}$. Now let $\mathscr{N}$ denote any complementary sub-

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bundle of $\mathscr{H}$ in $\mathscr{E}$. For example if one uses the Riemannian structure on M, the subbundle $\mathscr{N}$ can be chosen to be the orthogonal complement $\mathscr{H}^{\perp}$. The flow π then induces a flow π_{N} on $\mathscr{N}$. We shall let π_{T} denote the restriction of π to $\mathscr{H}$ and π_{T} is called the tangential part of π. Similarly the flow π_{N} is called the normal part of π. Each of the three linear skew-product flows π, π_{T} and π_{N} has a spectrum Σ, Σ_{T} and Σ_{N} which we describe in Section IV. We will show that none of these three spectra depend on the choice of the complement $\mathscr{N}$. In Section V we will show that in general one has $\Sigma \subseteq \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{N}}$. Furthermore if the flow on Y is chain recurrent, or if $\Sigma_{\mathrm{T}} \cap \Sigma_{\mathrm{N}} = \varnothing$, then one can show that $\Sigma = \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{N}}$. Finally we show that if $\Sigma_{\mathrm{T}} \cap \Sigma_{\mathrm{N}} = \varnothing$, then the flow π has an *invariant* complementary subbundle.

These relationships are especially useful because it is known that each of the three spectra is simply the union of a finite number of closed non-overlapping intervals, [26]. Furthermore associated with each spectral interval is a nontrivial invariant subbundle in the respective bundle $\mathscr{E}$, $\mathscr{H}$ or $\mathscr{N}$. In our main result (Theorem 1) we also show a simple relationship between these invariant subbundles for the flow π on $\mathscr{E}$ and those generated by π_{T} and π_{N}.

The study of exponential growth rates of solutions has a long history in the theory of differential equations, going back to Liapunov. As a result one can find precursors of our theory in many papers. A representative, but not complete, list is [1–34]. Additional references can be found in [3, 4, 12, 31]. Some theories concerning the perturbation of invariant manifolds [12, 15, 23] and some recent results in bifurcation theory [30] can be formulated in terms of this spectral theory.

As mentioned above, we shall develop our theory here for linear skew-product flows on vector bundles. In this way our theory not only applies to the study of invariant submanifolds of a flow, but also to block triangular differential systems of the form

$$\begin{aligned} u' &= A(t)u + B(t)v, \\ v' &= \phantom{A(t)u + {}} D(t)v. \end{aligned}$$

This application is discussed in Sections VII and VIII, where it is shown that the equality $\Sigma = \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{N}}$ can fail to hold when the normal and tangential spectra are not disjoint and the associated flow on Y is not chain recurrent.

II. Linear Skew-Product Flows

Let $\mathscr{E}$ be a vector bundle over a base space Y with projection p. This means that $\mathscr{E}$ and Y are topological spaces and that the following hold:

(1) $p: \mathscr{E} \to Y$ is a continuous mapping of $\mathscr{E}$ onto Y.

(2) For each $y \in Y$, the fibre $\mathscr{E}(y) = p^{-1}(y)$ is a vector space.

(3) For each $y \in Y$, there is an open set U containing y, a vector space X and a homeomorphism

$$\tau: p^{-1}(U) \to X \times U$$

of $p^{-1}(U)$ onto $X \times U$, which can be represented in the form

$$\tau(x, \eta) = (\tau_\eta x, \eta),$$

where τ_η is a linear isomorphism of the fibre $\mathscr{E}(y)$ onto X. (Here we assume that X is finite dimensional, i.e., $X = R^n$, or C^n, and that Y is a metric space.)

Notational convention. In statement (3) we adopt a notational convention which is convenient for many purposes. A point in $\mathscr{E}$ is denoted by an ordered pair (x, y) where $y \in Y$ and x is a vector in the fibre $\mathscr{E}(y)$. In this case the projection p is simply $p(x, y) = y$. Statement (3) is then the precise way of starting that a vector bundle $\mathscr{E}$ is "locally" a product space.

Let $\mathscr{E}$ be a vector bundle over a base space Y with projection p. A subset $\mathscr{H} \subseteq \mathscr{E}$ is said to be a ***subbundle*** of $\mathscr{E}$ if $\mathscr{H}$ itself is a vector bundle over Y with projection map $p \mid \mathscr{H}$ = restriction of p to $\mathscr{H}$. One can show that $\mathscr{H} \subseteq \mathscr{E}$ is a subbundle of $\mathscr{E}$ if and only if the following three conditions hold:

(1) $\mathscr{H}$ is a closed subset of $\mathscr{E}$.

(2) For each $y \in Y$ the fibre

$$\mathscr{H}(y) = \{x \in \mathscr{E}(y): (x, y) \in \mathscr{H}\}$$

is a linear subspace of $\mathscr{E}(y)$.

(3) The function $\dim \mathscr{H}(y)$ is constant on each component of Y; cf. [28].

The simplest vector bundle is a topological product space $\mathscr{E} = X \times Y$ where X is a (finite dimensional) linear space and Y is a metric space. A more complicated example occurs when Y is a smooth submanifold of a smooth manifold M, both without boundaries. Let TM be the tangent bundle of M, that is, TM is the usual union of the tangent spaces $T_yM = TM(y)$ and let $p: TM \to M$ be the projection that maps each $TM(y)$ onto the base point y. Let

$$\mathscr{E} = \bigcup_{y \in Y} TM(y)$$

be the restriction of the tangent bundle to the base space Y and consider the subbundle of $\mathscr{E}$ given by $\mathscr{H} = TY$ = tangent bundle of Y. Let $\mathscr{N}$ be a subbundle of $\mathscr{E}$ that is complementary to $\mathscr{H}$. Then $\mathscr{E} = \mathscr{H} + \mathscr{N}$, as a Whitney sum, i.e.,

(1) $\mathscr{N}(y) \cap \mathscr{H}(y) = \{0\}$ for all $y \in Y$,

(2) $\mathscr{E}(y) = \mathscr{H}(y) + \mathscr{N}(y)$ for all $y \in Y$.

This complementary bundle $\mathscr{N}$ is not unique (when $\mathscr{H} \neq \mathscr{E}, \mathscr{E}_0$) and we shall discuss this point further.

In our definition of a vector bundle we do allow the dimension of the fibres $\mathscr{E}(y)$ to vary. However, it is easily seen that, over each component of Y, the function $\dim \mathscr{E}(y)$ is constant. The dimension of a vector bundle $\dim \mathscr{E}$ is defined to be the $\max\{\dim \mathscr{E}(y): y \in Y\}$ whenever this maximum exits. When the base space Y is connected, one then has $\dim \mathscr{E} = \dim \mathscr{E}(y)$ for all $y \in Y$.

Let $\mathscr{E}$ be a vector bundle over the base space Y with projection map p. A mapping $\hat{P}: \mathscr{E} \to \mathscr{E}$ is said to be a *projector* if

(1) $\hat{P}$ is continuous, and

(2) for each $y \in Y$, there is a linear projection $P(y)$ in $\mathscr{E}(y)$ such that

$$\hat{P}(x, y) = (P(y)x, y)$$

for all $(x, y) \in \mathscr{E}$. In other words, $\hat{P}$ agrees with $P(y)$ in the fibre $\mathscr{E}(y)$, and it is a linear projection. If $\hat{P}$ is a projector on $\mathscr{E}$, then the

$$\text{range } \hat{P} = \mathscr{R} = \{(x, y) \in \mathscr{E}: P(y)x = x\},$$
$$\text{null space } \hat{P} = \mathscr{N} = \{(x, y) \in \mathscr{E}: P(y)x = 0\}$$

are complementary subbundles of $\mathscr{E}$, that is, $\mathscr{R}(y) \cap \mathscr{N}(y) = \{0\}$ and $\mathscr{E}(y) = \mathscr{R}(y) + \mathscr{N}(y)$ for all $y \in Y$. Conversely if $\mathscr{H}$ and $\mathscr{V}$ are two complementary subbundles in $\mathscr{E}$, then there is a unique projector $\hat{P}$ on $\mathscr{E}$ such that $\mathscr{R} = \mathscr{V}$ and $\mathscr{N} = \mathscr{H}$. See [28] for more details.

Flows. Let W be a topological space and let R denote the real numbers. A *flow* on W is a continuous mapping $\pi: W \times R \to W$ that satisfies

(1) $\pi(w, 0) = w$,

(2) $\pi(\pi(w, s), t) = \pi(w, s + t)$

for all $w \in W$ and $s, t \in R$.

Let π be a flow on W and σ a flow on Y. A mapping $q: W \to Y$ is said to be a *homomorphism* if

(1) q is a continuous mapping of W onto Y, and

(2) q commutes with the flows, i.e.,

$$q(\pi(w, t)) = \sigma(q(w), t)$$

for all $w \in W$ and $t \in R$. (We shall denote this phenomenon by writing $q: (W, \pi) \to (Y, \sigma)$ is a homomorphism.)

Let π be a flow on a vector bundle $\mathscr{E}$ with base space Y and projection p. The flow π is said to be a *skew-product flow* on $\mathscr{E}$ if there is a flow σ on Y such that the projection $p: (\mathscr{E}, \pi) \to (Y, \sigma)$ is a homomorphism. It is convenient to use the notational convention introduced above where $p(x, y) = y$. Let $(x, y) \in \mathscr{E}$ and let σ denote the flow on Y. Then $p: (\mathscr{E}, \pi) \to (Y, \sigma)$ is a homomorphism if and only if π can be represented in the form $\pi = (\varphi, \sigma)$ where

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t))$$

and $\varphi(x, y, t)$ is in the fibre $\mathscr{E}(\sigma(y, t))$. Let us abbreviate $\sigma(y, t) = y \cdot t$. This skew-product flow π is a *linear skew-product flow* if in addition the mapping

$$x \to \Phi(y, t)x = \varphi(x, y, t)$$

is a linear mapping from the fibre $\mathscr{E}(y)$ to the fibre $\mathscr{E}(y \cdot t)$.

A skew-product flow $\pi = (\varphi, \sigma)$ is said to be *smooth* if for each $(x, y) \in \mathscr{E}$, the function $\varphi(x, y, t)$ is differentiable at $t = 0$ and the mapping

$$(x, y) \to \left(\frac{\partial}{\partial t} \varphi(x, y, t) \big|_{t=0}, y\right)$$

is a continuous mapping of $\mathscr{E}$ into itself. (Note that we do not require $\sigma(y, t)$ to be differentiable. This is important for some applications.)

III. Local Coordinates Induced by an Invariant Subbundle

Let $\pi = (\varphi, \sigma)$ be a linear skew-product flow on a vector bundle $\mathscr{E}$ over a base space Y. Let $\mathscr{H}$ be an invariant subbundle. Let $\mathscr{N}$ denote any complementary subbundle of $\mathscr{H}$ in $\mathscr{E}$ and let $\hat{P}$ denote the unique projector on $\mathscr{E}$ with range $= \mathscr{N}$ and null space $= \mathscr{H}$. In this section we shall describe a suitable system of *local coordinates* induced by $\mathscr{H}$ (and $\mathscr{N}$), and we will show how the flow π can be represented in terms of these coordinates by using an appropriate variation of constants formula. In the case that $\mathscr{E} = X \times Y$ is a product space, the proofs can be found in [25, Part III]. However, since the proofs carry over, almost verbatim, to the general vector bundle $\mathscr{E}$, we shall omit the detailed arguments.

For $(x, y) \in \mathscr{N}$ we define $\pi_N(x, y, t)$ by

$$\pi_N(x, y, t) = \hat{P}(\pi(x, y, t)) = (P(y \cdot t)\, \varphi(x, y, t), y \cdot t).$$

Since $\mathscr{H}$ is an invariant set for π, the mapping π_N is a linear skew-product flow on $\mathscr{N}$. Let φ_N and Φ_N be defined by

$$\varphi_N(x, y, t) = P(y \cdot t)\, \varphi(x, y, t); \qquad \Phi_N(y, t) = P(y \cdot t)\, \Phi(y, t).$$

Then the flow π_N, which we shall call the *normal flow* induced by π on $\mathcal{N}$, can be represented by

$$\pi_N(x, y, t) = (\varphi_N(x, y, t), y \cdot t) = (\Phi_N(y, t)x, y \cdot t).$$

The restriction of the flow π to the invariant subbundle $\mathcal{H}$ defines a flow on $\mathcal{H}$. We shall denote this restriction by π_T and we shall call it the *tangential flow* induced by π on $\mathcal{H}$. This flow π_T can be represented as

$$\pi_T(x, y, t) = (\varphi_T(x, y, t), y \cdot t) = (\Phi_T(y, t)x, y \cdot t),$$

where $\varphi_T(x, y, t) = \varphi(x, y, t)$ for $(x, y) \in \mathcal{H}$. The linear transformation $\Phi_T(y, t)$ agrees with $\Phi(y, t)$ on $\mathcal{H}(y)$, i.e. $\Phi_T(y, t) = \Phi(y, t) \mid \mathcal{H}(y)$.

Now let $(x, y) \in \mathcal{E}$ and define $v = P(y)x$ and $u = x - P(y)x$. Then

$$(x, y) = (u + v, y) = (u, y) + (v, y)$$

is the unique decomposition of (x, y) into the sum of two vectors $(u, y) \in \mathcal{H}$ and $(v, y) \in \mathcal{N}$. The corresponding solutions $\varphi(u, y, t)$ and $\varphi(v, y, t)$ admit similar decompositions. Since $\mathcal{H}$ is invariant under π one has $(\varphi(u, y, t), y \cdot t) \in \mathcal{H}$ for all $t \in R$, whenever $(u, y) \in \mathcal{H}$. This means that one can write

$$\varphi(u, y, t) = A(y, t)u,$$

where $A(y, t)$ is a linear transformation from $\mathcal{H}(y)$ to $\mathcal{H}(y \cdot t)$. Clearly one must have $A(y, t) = \Phi_T(y, t)$. Similarly one has

$$\varphi(v, y, t) = B(y, t)v + D(y, t)v,$$

where $B(y, t): \mathcal{N}(y) \to \mathcal{H}(y \cdot t)$ and $D(y, t): \mathcal{N}(y) \to \mathcal{N}(y \cdot t)$ are linear transformations. Then

$$\begin{aligned} P(y \cdot t)\, \varphi(v, y, t) &= P(y \cdot t)\, B(y, t)v + P(y \cdot t)\, D(y, t)v \\ &= 0 + D(y, t)v \end{aligned}$$

since $B(y, t)v \in \text{Null space } (P(y \cdot t))$ and $D(y, t)v \in \text{Range}(P(y \cdot t))$. Hence one has $D(y, t) = \Phi_N(y, t)$.

If one writes x in the vector form $x = \binom{u}{v}$, then the above paragraph can be summarized by using matrix notation and writing

$$\begin{aligned} \varphi(x, y, t) &= \Phi(y, t)x = \Phi(y, t)\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} A(y, t)u + B(y, t)v \\ D(y, t)v \end{pmatrix} = \begin{pmatrix} A(y, t) & B(y, t) \\ 0 & D(y, t) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \end{aligned}$$

In other words, one has the local representation

$$\Phi(y, t) = \begin{pmatrix} A(y, t) & B(y, t) \\ 0 & D(y, t) \end{pmatrix}, \tag{3.1}$$

where $\Phi_{\mathrm{T}}(y, t) = A(y, t)$ and $\Phi_{\mathrm{N}}(y, t) = D(y, t)$.

We shall call any linear skew-product flow π on a vector bundle $\mathscr{E}$ that can be represented in the form (3.1), for some choice of local coordinates $\binom{u}{v}$, a *block triangular system*. It follows then that every invariant subbundle $\mathscr{H}$ gives rise to a block triangular system (3.1) where $u \in \mathscr{H}(y)$, $v \in \mathscr{N}(y)$ and $\mathscr{N}$ is some complementary bundle of $\mathscr{H}$ in $\mathscr{E}$. Moreover the converse is also true. If there is a projector $\hat{P}$ on $\mathscr{E}$ such that, in terms of the local coordinates $u \in \mathscr{R}(y)$ and $v \in \mathscr{N}(y)$, where $\mathscr{R}$ and $\mathscr{N}$ denote the range and null space of $\hat{P}$, the flow π can be represented as a block triangular system (3.1), then the null space $\mathscr{N}$ is an invariant subbundle for π and the range $\mathscr{R}$ is a complementary subbundle.

If $\Phi(y, t)$ can be represented in the form (3.1) where the linear transformation B vanishes, i.e., if $\Phi(y, t)$ has the form

$$\Phi(y, t) = \begin{pmatrix} A(y, t) & 0 \\ 0 & D(y, t) \end{pmatrix}, \tag{3.2}$$

where $u \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$, then we shall call the flow π a *block diagonal system* with respect to the local coordinates $\binom{u}{v}$. The following proposition is now immediate:

Lemma 1. *Let $\mathscr{H}$ and $\mathscr{N}$ be complementary subbundles in $\mathscr{E}$ and let π be a linear skew-product flow on $\mathscr{E}$. Then π is a block diagonal system with respect to coordinates $u \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$ if and only if both $\mathscr{H}$ and $\mathscr{N}$ are invariant sets for π. Furthermore, in this case, the induced normal flow π_{N} on $\mathscr{N}$ agrees with the flow π.*

Next let us assume that π is a smooth linear skew-product flow on a vector bundle $\mathscr{E}$ with an invariant subbundle $\mathscr{H}$ and complementary bundle $\mathscr{N}$. Also let (3.1) be the representation of π as a block triangular system. Since π is smooth, the derivative

$$b(y) = \frac{\partial}{\partial t} B(y, t)\bigg|_{t=0} \tag{3.3}$$

exists, and $b(y)$ is a linear transformation from $\mathscr{N}(y)$ to $\mathscr{H}(y)$. Furthermore, the mapping $\hat{b}\colon \mathscr{N} \to \mathscr{H}$ defined by

$$\hat{b}(x, y) = (b(y)x, y)$$

is continuous. The following lemma is now immediate:

LEMMA 2. *Let π be a smooth linear skew-product flow on a vector bundle $\mathscr{E}$ with compact base space Y. Let $\mathscr{H}$ be an invariant subbundle of π of let $\mathscr{N}$ be a complementary subbundle. Let $b(y)$ be given by* (3.3). *Then there is a real constant M satisfying $|b(y)| \leqslant M$ for all $y \in Y$.*

Let π be a linear skew-product flow on a vector bundle $\mathscr{E}$. For each $\lambda \in R$ define π_λ by

$$\pi_\lambda(x, y, t) = (e^{-\lambda t}\varphi(x, y, t), y \cdot t)$$

Then it is easily seen that π_λ is also a linear skew-product flow on $\mathscr{E}$. Furthermore any subbundle $\mathscr{H}$ in $\mathscr{E}$ is π-invariant if and only if it is π_λ-invariant for every $\lambda \in R$. Define Φ_λ by

$$\Phi_\lambda(y, t)x = e^{-\lambda t}\varphi(x, y, t) = e^{-\lambda t}\Phi(y, t)x.$$

Assume that $\mathscr{H}$ is an invariant subbundle of π and let $\mathscr{N}$ be a complementary subbundle. Let (3.1) be the representation of π as a block triangular system and define A_λ, B_λ and D_λ by

$$A_\lambda(y, t) = e^{-\lambda t}A(y, t), \qquad B_\lambda(y, t) = e^{-\lambda t}B(y, t)$$
$$D_\lambda(y, t) = e^{-\lambda t}D(y, t).$$

Then it is easily seen that the representation of π_λ as a block triangular system is given by

$$\Phi_\lambda(y, t) = \begin{pmatrix} A_\lambda(y, t) & B_\lambda(y, t) \\ 0 & D_\lambda(y, t) \end{pmatrix}.$$

Furthermore, if in addition, π is a smooth linear skew-product flow, then the function $b(y)$ given by (3.3) also satisfies

$$b(y) = \frac{\partial}{\partial t} B_\lambda(y, t) \Big|_{t=0} \tag{3.4}$$

for every $\lambda \in R$, since $B(y, 0) = 0$ for all $y \in Y$.

The following lemma is proved in [25, Part III].

LEMMA 3. *Let π be a smooth linear skew product flow on a vector bundle $\mathscr{E}$ and let $\mathscr{H}$ be an invariant subbundle with complementary subbundle $\mathscr{N}$. Let* (3.1) *be the representation of π as a block triangular system. For $x \in \mathscr{E}(y)$ let $x = u + v$ where $u \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$. Furthermore let u_t and v_t be respectively the u- and v-coordinates of the time evolution of the solution*

$$\Phi_\lambda(y, t)x = e^{-\lambda t}\varphi(x, y, t) = u_t + v_t,$$

where $u_t \in \mathscr{H}(y \cdot t)$ *and* $v_t \in \mathscr{N}(y \cdot t)$. *Then one has*

$$\begin{aligned} u_t &= A_\lambda(y, t)\left[u + \int_0^t A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s) v\, ds\right], \\ v_t &= D_\lambda(y, t) v. \end{aligned} \tag{3.5}$$

IV. The Spectrum

Let $\pi = (\varphi, \sigma)$ be a linear skew-product flow on a vector bundle $\mathscr{E}$ with base space Y. We shall assume for the remainder of this section that Y is *compact and connected*. As defined above, π_λ is the linear skew-product flow given by

$$\pi_\lambda(x, y, t) = (\Phi_\lambda(y, t)x, y \cdot t) = (e^{-\lambda t}\varphi(x, y, t), y \cdot t),$$

where $\lambda \in R$. We shall say that the linear skew-product flow π_λ admits an *exponential dichotomy on* Y if there is a projector $\hat{P} = \hat{P}_\lambda$ on $\mathscr{E}$ and positive constants K and α such that

$$\begin{aligned} |\,\Phi_\lambda(y, t)\, P_\lambda(y)\, \Phi_\lambda^{-1}(y, s)| &\leqslant K e^{-\alpha(t-s)}, && s \leqslant t, \\ |\,\Phi_\lambda(y, t)[I - P_\lambda(y)]\, \Phi_\lambda^{-1}(y, s)| &\leqslant K e^{-\alpha(s-t)}, && t \leqslant s, \end{aligned}$$

for all $y \in Y$. In this case, the range of $\hat{P}_\lambda$ is precisely the stable set $\mathscr{S}_\lambda$, where $\mathscr{S}_\lambda$ is defined by

$$\mathscr{S}_\lambda = \{(x, y) \in \mathscr{E}: |\,\Phi_\lambda(y, t)x\,| \to 0 \text{ as } t \to +\infty\},$$

and the null space of $\hat{P}_\lambda$ is the unstable set $\mathscr{U}_\lambda$, where $\mathscr{U}_\lambda$ is defined by

$$\mathscr{U}_\lambda = \{(x, y) \in \mathscr{E}: |\,\Phi_\lambda(y, t)x\,| \to 0 \text{ as } t \to -\infty\}.$$

When π_λ admits an exponential dichotomy on Y, the sets $\mathscr{S}_\lambda$ and $\mathscr{U}_\lambda$ are complementary subbundles of $\mathscr{E}$. Whether π_λ admits a dichotomy or not, the sets $\mathscr{S}_\lambda$ and $\mathscr{U}_\lambda$ are invariant sets for π and for π_μ, for every $\mu \in R$.

The collection of all $\lambda \in R$ for which π_λ admits an exponential dichotomy on Y is called the *resolvent set of* π and is denoted by $\rho(\mathscr{E}, \pi)$. The complement of $\rho(\mathscr{E}, \pi)$ is called the *spectrum of* π *on* $\mathscr{E}$ and we denote this set by $\Sigma(\mathscr{E}, \pi)$. The following Spectral Theorem is proved in [26]:

Theorem A. *Let* $\pi = (\varphi, \sigma)$ *be a linear skew-product flow on a vector bundle* $\mathscr{E}$ *over a compact connected base space* Y. *Assume that* $\dim \mathscr{E} = n \geqslant 1$. *Then the spectrum* $\Sigma(\mathscr{E}, \pi)$ *is the union of k nonoverlapping nonempty compact intervals*

$$\Sigma(\mathscr{E}, \pi) = \bigcup_{i=1}^{k} [a_i, b_i],$$

where $1 \leqslant k \leqslant n$. *Furthermore associated with each spectral interval* $[a_i, b_i]$ *there is an invariant subbundle* $\mathscr{V}_i$ *of* $\mathscr{E}$ *with* $\dim \mathscr{V}_i = n_i$ *where*

(1) $1 \leqslant n_i$ *for* $1 \leqslant i \leqslant k$ *and* $n_1 + \cdots + n_k = n$,

(2) $\mathscr{V}_i(y) \cap \mathscr{V}_j(y) = \{0\}$ *for all* $y \in Y$ *when* $i \neq j$,

(3) $\mathscr{E}(y) = \mathscr{V}_1(y) + \cdots + \mathscr{V}_k(y)$ *for all* $y \in Y$,

(4) $\mathscr{V}_i = \mathscr{S}_\lambda \cap \mathscr{U}_\mu$ *whenever* $\mu, \lambda \in \rho(\mathscr{E}, \pi)$

and $(\mu, \lambda) \cap \Sigma(\mathscr{E}, \pi) = [a_i, b_i]$.

The invariant subbundle $\mathscr{V}_i$ is called the spectral subbundle associated with the spectral interval $[a_i, b_i]$. As a result of statement (4), we see that for each $y \in Y$ the linear space $\mathscr{V}_i(y)$ consists of those vectors $x \in \mathscr{E}(y)$ for which $x = 0$, or (when $x \neq 0$) for which lim sup, or lim inf, as $t \to +\infty$, or $-\infty$, of

$$t^{-1} \log |\varphi(x, y, t)|$$

lie in the interval $[a_i, b_i]$, i.e., the growth rates of $\varphi(x, y, t)$ lie in the spectral interval $[a_i, b_i]$.

We shall require some necessary and sufficient conditions for an exponential dichotomy later. In order to describe these conditions we first define the *bounded set*

$$\mathscr{B}_\lambda = \{(x, y) \in \mathscr{E} : \sup_{t \in R} |\Phi_\lambda(y, t)x| < \infty\}$$

and the *zero section* $\mathscr{E}_0 = \{(x, y) \in \mathscr{E} : x = 0\}$.

We shall need the concept of chain recurrence, which is defined in [6, 26]. (Also see the notion of pseudo-orbit in [36].) The following theorem is proved in [25]:

THEOREM B. *Let* $\pi = (\varphi, \sigma)$ *be a linear skew-product flow on a vector bundle* $\mathscr{E}$ *with a compact connected base space* Y *and let* M *denote the union of the minimal sets in* Y. *Let* $\lambda \in R$. *Then the following statements are valid*:

(1) *If* $\lambda \in \rho(\mathscr{E}, \pi)$, *then* $\mathscr{B}_\lambda = \mathscr{E}_0$

(2) *If* $\mathscr{B}_\lambda = \mathscr{E}_0$ *and* $\dim \mathscr{S}_\lambda(y)$ *is constant for* $y \in M$, *then* $\lambda \in \rho(\mathscr{E}, \pi)$.

(3) *If* $\mathscr{B}_\lambda = \mathscr{E}_0$ *and the flow* σ *on* Y *is chain-recurrent*, *then* $\lambda \in \rho(\mathscr{E}, \pi)$.

In particular if $\mathscr{B}_\lambda = \mathscr{E}_0$ *and* Y *is a minimal set then* $\lambda \in \rho(\mathscr{E}, \pi)$.

V. The Spectra of an Invariant Subbundle

Let $\pi = (\varphi, \sigma)$ be a linear skew-product flow on a vector bundle $\mathscr{E}$ with base space Y, and assume that Y is a compact connected metric space. Let $\mathscr{H}$ be an invariant subbundle of π and let $\mathscr{N}$ denote a complementary subbundle. Let π_T and π_N denote the tangential and normal flows induced by π on $\mathscr{H}$ and $\mathscr{N}$, respectively. The Spectral Theorem A applies to each of the flows, $(\mathscr{E}, \pi)$, $(\mathscr{H}, \pi_T)$ and $(\mathscr{N}, \pi_N)$. In particular each of the spectra $\Sigma(\mathscr{E}, \pi)$, $\Sigma(\mathscr{H}, \pi_T)$ and $\Sigma(\mathscr{N}, \pi_N)$ is the finite union of nonoverlapping compact intervals and the number of such intervals is bounded, respectively, by $\dim \mathscr{E}$, $\dim \mathscr{H}$ and $\dim \mathscr{N}$.

Even though the given subbundle $\mathscr{H}$ may have many complementary subbundles $\mathscr{N}$, the following result shows that the spectrum $\Sigma(\mathscr{N}, \pi_N)$ does not depend on the choice of the subbundle $\mathscr{N}$.

LEMMA 4. *Let $\mathscr{N}$ and $\mathscr{M}$ be two complementary subbundles to $\mathscr{H}$ and let π_N and π_M denote the corresponding induced flows on $\mathscr{N}$ and $\mathscr{M}$, respectively. Then $\Sigma(\mathscr{N}, \pi_N) = \Sigma(\mathscr{M}, \pi_M)$.*

Proof. Let $\hat{P}_N$ and $\hat{P}_M$ denote the projectors on $\mathscr{E}$ with range $\hat{P}_N = \mathscr{N}$, range $\hat{P}_M = \mathscr{M}$ and

$$\text{null space } \hat{P}_N = \text{null space } \hat{P}_M = \mathscr{H}.$$

Since $\mathscr{M}$ and $\mathscr{N}$ are both complementary to $\mathscr{H}$ it follows that the restrictions $\hat{P}_M \mid \mathscr{N}$ and $\hat{P}_N \mid \mathscr{M}$ are bundle isomorphisms. Consequently one has $\hat{P}_M\hat{P}_N = \hat{P}_M$ and $\hat{P}_N\hat{P}_M = \hat{P}_N$. Furthermore these restrictions commute with the induced flows π_N and π_M. Indeed if $x \in \mathscr{N}(y)$ and $u = x - P_M(y)x$ then $u \in \mathscr{H}(y)$. Also one has

$$\begin{aligned} P_M(y \cdot t)\, \varphi_N(x, y, t) &= P_M(y \cdot t)\, P_N(y \cdot t)\, \varphi(x, y, t) \\ &= P_M(y \cdot t)\, P_N(y \cdot t)[\varphi(P_M(y)x, y, t) + \varphi(u, y, t)] \\ &= P_M(y \cdot t)\, P_N(y \cdot t)\, \varphi(P_M(y)x, y, t) + 0, \end{aligned}$$

since $\mathscr{H}$ is invariant. Since $P_M P_N = P_M$ one has

$$\begin{aligned} P_M(y \cdot t)\, \varphi_N(x, y, t) &= P_M(y \cdot t)\, \varphi(P_M(y)x, y, t) \\ &= \varphi_M(P_M(y)x, y, t). \end{aligned}$$

This proves that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{N} \times R & \xrightarrow{\pi_N} & \mathscr{N} \times R \\ {\scriptstyle P_M\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle P_M\times \mathrm{id}} \\ \mathscr{M} \times R & \xrightarrow[\pi_M]{} & \mathscr{M} \times R \end{array}$$

Since $\hat{P}_M \mid \mathscr{N} : \mathscr{N} \to \mathscr{M}$ is a flow homomorphism it follows immediately that $\Sigma(\mathscr{N}, \pi_N) = \Sigma(\mathscr{M}, \pi_M)$. We omit these details. Q.E.D.

Throughout the remainder of this section we shall make the following

STANDING HYPOTHESIS. *Let $\pi = (\varphi, \sigma)$ be a smooth linear skew-product flow on a vector bundle $\mathscr{E}$ with compact, connected metrizable base space Y. Let $\mathscr{H}$ be an invariant subbundle for π and let $\mathscr{N}$ be a fixed complementary subbundle to $\mathscr{H}$. Let π_T and π_N denote the tangential and normal flows induced by π on $\mathscr{H}$ and $\mathscr{N}$, respectively, and let $\Sigma(\mathscr{E}, \pi)$, $\Sigma(\mathscr{H}, \pi_T)$ and $\Sigma(\mathscr{N}, \pi_N)$ denote the spectra of these three flows.*

For $\lambda \in R$ we let $\pi_{T,\lambda}$ and $\pi_{N,\lambda}$ denote the respective flows

$$\pi_{T,\lambda}(x, y, t) = (e^{-\lambda t}\Phi_T(y, t)x, y \cdot t) = (A_\lambda(y, t)x, y \cdot t),$$
$$\pi_{N,\lambda}(x, y, t) = (e^{-\lambda t}\Phi_N(y, t)x, y \cdot t) = (D_\lambda(y, t)x, y \cdot t)$$

on $\mathscr{H}$ and $\mathscr{N}$, where $A_\lambda(y, t)$ and $D_\lambda(y, t)$ are defined in Section III. Also $\mathscr{S}_{T,\lambda}$, $\mathscr{U}_{T,\lambda}$, $\mathscr{B}_{T,\lambda}$, $\mathscr{S}_{N,\lambda}$, $\mathscr{U}_{N,\lambda}$ and $\mathscr{B}_{N,\lambda}$ will denote the respective stable set, unstable set and bounded set for $\pi_{T,\lambda}$ and $\pi_{N,\lambda}$. We can now prove the following result:

THEOREM 1. *Under the Standing Hypothesis one has*

$$\Sigma(\mathscr{E}, \pi) \subseteq \Sigma(\mathscr{H}, \pi_T) \cup \Sigma(\mathscr{N}, \pi_N).$$

Moreover, if $\lambda \in \rho(\mathscr{H}, \pi_T) \cap \rho(\mathscr{N}, \pi_N)$, then $\lambda \in \rho(\mathscr{E}, \pi)$ and

$$\dim \mathscr{S}_\lambda = \dim \mathscr{S}_{T,\lambda} + \dim \mathscr{S}_{N,\lambda}$$
$$\dim \mathscr{U}_\lambda = \dim \mathscr{U}_{T,\lambda} + \dim \mathscr{U}_{N,\lambda}$$

Proof. Let $\lambda \in \rho(\mathscr{H}, \pi_T) \cap \rho(\mathscr{N}, \pi_N)$. Then the two flows $\pi_{T,\lambda}$ and $\pi_{N,\lambda}$ admit exponential dichotomies on Y. This means that there are projectors $\hat{P}_{T,\lambda}$ and $\hat{P}_{N,\lambda}$ on $\mathscr{H}$ and $\mathscr{N}$ and positive constants K and α such that the following inequalities hold:

$$| A_\lambda(y, t)\, P_{T,\lambda}(y)\, A_\lambda^{-1}(y, s)| \leqslant Ke^{-\alpha(t-s)}, \qquad s \leqslant t, \tag{5.1}$$
$$| A_\lambda(y, t)[I_T - P_{T,\lambda}(y)]\, A_\lambda^{-1}(y, s)| \leqslant Ke^{-\alpha(s-t)}, \qquad t \leqslant s, \tag{5.2}$$
$$| D_\lambda(y, t)\, P_{N,\lambda}(y)\, D_\lambda^{-1}(y, s)| \leqslant Ke^{-\alpha(t-s)}, \qquad s \leqslant t, \tag{5.3}$$
$$| D_\lambda(y, t)[I_N - P_{N,\lambda}(y)]\, D_\lambda^{-1}(y, s)| \leqslant Ke^{-\alpha(s-t)}, \qquad t \leqslant s, \tag{5.4}$$

for all $y \in Y$, where I_T and I_N denote the identity maps on $\mathscr{H}(y)$ and $\mathscr{N}(y)$, respectively. Recall that

$$\mathscr{S}_{N,\lambda} = \text{range } \hat{P}_{N,\lambda}, \qquad \mathscr{U}_{N,\lambda} = \text{null space } \hat{P}_{N,\lambda},$$
$$\mathscr{S}_{T,\lambda} = \text{range } \hat{P}_{T,\lambda}, \qquad \mathscr{U}_{T,\lambda} = \text{null space } \hat{P}_{T,\lambda}.$$

Let b be given by (3.3) and (3.4) and define the transformation $\hat{L}$ on $\mathscr{S}_{N,\lambda}$ formally by $\hat{L}(x, y) = (L(y)x, y)$, where

$$L(y)x = -\int_0^\infty [I_T - P_{T,\lambda}(y)]\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)x\, ds, \tag{5.5}$$

where $(x, y) \in \mathscr{S}_{N,\lambda}$. We will now show that $\hat{L}$ maps $\mathscr{S}_{N,\lambda}$ into $\mathscr{H}$. First note that since $(x, y) \in \mathscr{S}_{N,\lambda}$ one has

$$| D_\lambda(y, s)x | \leqslant K \,| x |\, e^{-\alpha s}, \qquad s \geqslant 0,$$

by (5.3). Next observe that $| b(y \cdot s)| \leqslant M$ for $s \geqslant 0$ by Lemma 2. Furthermore (5.2) implies that

$$|[I_T - P_{T,\lambda}(y)]\, A_\lambda^{-1}(y, s)| \leqslant Ke^{-\alpha s}, \qquad s \geqslant 0,$$

and consequently the integral in (5.5) exists. Hence $\hat{L}$ is well defined on $\mathscr{S}_{N,\lambda}$ and (clearly) $L(y)x$ is linear in x. Since

$$L(y)x \in \text{range}([I_T - P_{T,\lambda}(y)]) = \text{null space}(P_{T,\lambda}(y))$$

it follows that $L(y)x \in \mathscr{U}_{T,\lambda}(y) \subseteq \mathscr{H}(y)$, i.e., $\hat{L}$ maps $\mathscr{S}_{N,\lambda}$ into $\mathscr{H}$.

Now choose $u \in \mathscr{S}_{T,\lambda}(y)$ and $v \in \mathscr{S}_{N,\lambda}(y)$ and with $L = (L)y$ set

$$\begin{aligned} u_t &= A_\lambda(y, t) \left[u + Lv + \int_0^t A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)\, v\, ds\right], \\ v_t &= D_\lambda(y, t)v. \end{aligned} \tag{5.6}$$

Since $(u + Lv) \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$, it follows from Lemma 3 that u_t and v_t represent the time-evolution of the u and v coordinates of the solution $e^{-\lambda t}\Phi(y, t)(u + Lv + v)$. In other words one has

$$e^{-\lambda t}\Phi(y, t)(u + Lv + v) = u_t + v_t\,, \qquad t \in R.$$

Since $v \in \mathscr{S}_{N,\lambda}(y)$, it follows from (5.3) that

$$| v_t | = | D_\lambda(y, t)v | \leqslant K \,| v |\, e^{-\alpha t}$$

for all $t \geqslant 0$. On the other hand, u_t can be rewritten as

$$\begin{aligned} u_t = A_\lambda(y, t)u &+ \int_0^t A_\lambda(y, t)\, P_{T,\lambda}(y)\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)v\, ds \\ &- \int_t^\infty A_\lambda(y, t)[I_T - P_{T,\lambda}(y)]\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)v\, ds. \end{aligned}$$

Since $| b(y \cdot s)\, D_\lambda(y, s)\, v | \leqslant KM | v | e^{-\alpha s}$ for $s \geqslant 0$, it follows from standard arguments [7, pp. 80–81] that $u_t \to 0$ as $t \to +\infty$. Hence $e^{-\lambda t}\Phi(y, t)(u + Lv + v) \to 0$ as $t \to +\infty$ and consequently

$$(u + Lv + v) \in \mathscr{S}_\lambda(y).$$

We have shown that the mapping

$$G(y)\colon (u, v) \to (u + Lv + v)$$

is a linear transformation of $\mathscr{S}_{\mathrm{T},\lambda}(y) \oplus \mathscr{S}_{\mathrm{N},\lambda}(y)$ into $\mathscr{S}_\lambda(y)$. Since $(u + L(y)v) \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$, we see that the mapping $G(y)$ is one-to-one. We will now show that range$(G(y)) = \mathscr{S}_\lambda(y)$. Let $(x, y) \in \mathscr{S}_\lambda$ and write $x = u + v$ where $u \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$. Then by Lemma 3 one has $e^{-\lambda t}\Phi(y, t)x = u_t + v_t$, where u_t and v_t satisfy (3.5). Since $| e^{-\lambda t}\Phi(y, t)x | \to 0$ as $t \to +\infty$, it follows from (3.5) that $| v_t | \to 0$ as $t \to +\infty$. Hence $v \in \mathscr{S}_{\mathrm{N},\lambda}(y)$ and by (5.3) one has

$$| v_t | = | D_\lambda(y, t)v | \leqslant K | v | e^{-\alpha t}, \qquad t \geqslant 0.$$

Now define $\tilde{u}$ by $\tilde{u} = u - L(y)v$. Then u_t becomes

$$\begin{aligned} u_t &= A_\lambda(y, t) \left[\tilde{u} + L(y)v + \int_0^t A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)v\, ds\right] \\ &= A_\lambda(y, t)\tilde{u} + \int_0^t A_\lambda(y, t)\, P_{\mathrm{T},\lambda}(y)\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)v\, ds \\ &\quad - \int_t^\infty A_\lambda(y, t)[I_{\mathrm{T}} - P_{\mathrm{T},\lambda}(y)]\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)v\, ds. \end{aligned}$$

Once again one has $| b(y \cdot s)\, D_\lambda(y, s)v | \leqslant KM | v | e^{-\alpha s}$ for $s \geqslant 0$ and consequently the two integrals above tend to 0 as $t \to +\infty$; cf. [7, pp. 80–81]. Since

$$| u_t | = | e^{-\lambda t}\Phi(y, t)x - v_t | \to 0 \qquad \text{as} \quad t \to +\infty,$$

it follows that $| A_\lambda(y, t)\tilde{u} | \to 0$ as $t \to +\infty$ and therefore $\tilde{u} \in \mathscr{S}_{\mathrm{T},\lambda}(y)$. Hence

$$u + v = \tilde{u} + L(y)v + v = G(y)(\tilde{u}, v),$$

i.e., $G(y)$ is surjective. This completes the proof that

$$\dim \mathscr{S}_\lambda = \dim \mathscr{S}_{\mathrm{T},\lambda} + \dim \mathscr{S}_{\mathrm{N},\lambda} \tag{5.7}$$

whenever $\lambda \in \rho(\mathscr{H}, \pi_{\mathrm{T}}) \cap \rho(\mathscr{N}, \pi_{\mathrm{N}})$.

A similar argument can be applied to study the connection between $\mathscr{U}_{\mathrm{T},\lambda}$,

$\mathscr{U}_{N,\lambda}$ and $\mathscr{U}_\lambda$. Without going into details, one shows that the mapping

$$H(y)\colon (u, v) \to u + M(y)v + v,$$

where $\hat{M}\colon \mathscr{U}_{N,\lambda} \to \mathscr{H}$ is given by $\hat{M}(v, y) = (M(y)v, y)$ and

$$M(y)v = \int_{-\infty}^{0} P_{T,\lambda}(y)\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)v\, ds,$$

is a linear isomorphism of $\mathscr{U}_{T,\lambda}(y) \oplus \mathscr{U}_{N,\lambda}(y)$ onto $\mathscr{U}_\lambda(y)$ for every $y \in Y$. Consequently for $\lambda \in \rho(\mathscr{H}, \pi_T) \cap \rho(\mathscr{N}, \pi_N)$ one has

$$\dim \mathscr{U}_\lambda = \dim \mathscr{U}_{T,\lambda} + \dim \mathscr{U}_{N,\lambda}. \tag{5.8}$$

Next we claim that $\mathscr{B}_\lambda = \mathscr{E}_0$. Indeed if $(x, y) \in \mathscr{B}_\lambda$ then the solution $e^{-\lambda t}\Phi(y, t)x$ is bounded for all $t \in R$. Let $x = u + v$ be the decomposition of x into vectors $u \in \mathscr{H}(y)$ and $v \in \mathscr{N}(y)$ and let $e^{-\lambda t}\Phi(y, t)x = u_t + v_t$ be the time-evolution of this solution given by Lemma 3. It then follows that both $|u_t|$ and $|v_t|$ are bounded for all $t \in R$. Since $\lambda \in \rho(\mathscr{N}, \pi_N)$ it follows from Theorem B(1) that $v = 0$. Consequently it follows from (3.5) that $u_t = A_\lambda(y, t)u$. Since $\lambda \in \rho(\mathscr{H}, \pi_T)$ it follows, again from Theorem B(1), that $u = 0$, whence $x = 0$ and $\mathscr{B}_\lambda = \mathscr{E}_0$. Next we note that (5.7) implies that $\dim \mathscr{S}_\lambda(y)$ is constant for $y \in Y$. Consequently Theorem B(2) implies that $\lambda \in \rho(\mathscr{E}, \pi)$. Q.E.D.

We will show below that the equality

$$\Sigma(\mathscr{E}, \pi) = \Sigma(\mathscr{H}, \pi_T) \cup \Sigma(\mathscr{N}, \pi_N) \tag{5.9}$$

can fail to hold. However, if the flow π has some additional structure one can prove that (5.9) is valid. Before doing this let us prove the following three lemmas:

LEMMA 5. *Assume that the Standing Hypothesis is satisfied. Then one has*

$$\lambda \in \rho(\mathscr{E}, \pi) \Rightarrow \mathscr{B}_{T,\lambda} = \mathscr{H}_0 \quad \textit{and} \quad \mathscr{B}_{N,\lambda} = \mathscr{N}_0.$$

Proof. Since $\pi_{T,\lambda}$ is the restriction of π_λ to $\mathscr{H}$ one has $\mathscr{B}_{T,\lambda} \subseteq \mathscr{B}_\lambda$. Since $\lambda \in \rho(\mathscr{E}, \pi)$ one has $\mathscr{B}_\lambda = \mathscr{E}_0$ by Theorem B(1). Hence $\mathscr{B}_{T,\lambda} = \mathscr{H}_0$.

In order to show that $\mathscr{B}_{N,\lambda} = \mathscr{N}_0$, we choose $(v_0, y) \in \mathscr{B}_{N,\lambda}$. Then $v_t = D_\lambda(y, t)\, v_0$ is bounded in t and (by Lemma 2) $b(y \cdot t)\, D_\lambda(y, t)\, v_0$ is also bounded in t. Next define u_t by

$$u_t = \int_{-\infty}^{t} A_\lambda(y, t)\, P_{T,\lambda}(y)\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)\, v_0\, ds - \int_{t}^{\infty} A_\lambda(y, t)[I_T - P_{T,\lambda}(y)]\, A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)\, v_0\, ds.$$

Because of (5.1) and (5.2) it is easy to verify that each of the above integrals exists and that the function u_t is bounded in t. Furthermore it also follows that u_t can be represented in the form

$$u_t = A_\lambda(y, t)\left[u_0 + \int_0^t A_\lambda^{-1}(y, s)\, b(y \cdot s)\, D_\lambda(y, s)\, v_0\, ds\right].$$

By Lemma 3 one has

$$\Phi_\lambda(y, t)(u_0 + v_0) = u_t + v_t,$$

and consequently the solution $\Phi_\lambda(y, t)(u_0 + v_0)$ is bounded in t. Since $\mathscr{B}_\lambda = \mathscr{E}_0$ one has $u_0 = v_0 = 0$. Hence $\mathscr{B}_{\mathrm{N},\lambda} = \mathscr{N}_0$. Q.E.D.

Lemma 6. *Assume that the Standing Hypothesis is satisfied. Then one has*

$$\Sigma(\mathscr{H}, \pi_{\mathrm{T}}) - \Sigma(\mathscr{E}, \pi) = \Sigma(\mathscr{N}, \pi_{\mathrm{N}}) - \Sigma(\mathscr{E}, \pi)$$

or equivalently

$$\rho(\mathscr{H}, \pi_{\mathrm{T}}) \cap \rho(\mathscr{E}, \pi) = \rho(\mathscr{N}, \pi_{\mathrm{N}}) \cap \rho(\mathscr{E}, \pi). \tag{5.10}$$

Proof. First assume that $\lambda \in \rho(\mathscr{H}, \pi_{\mathrm{T}}) \cap \rho(\mathscr{E}, \pi)$. Then the functions $\dim \mathscr{S}_\lambda(y)$ and $\dim \mathscr{S}_{\mathrm{T},\lambda}(y)$ are constant over Y. Furthermore one has $\mathscr{B}_{\mathrm{N},\lambda} = \mathscr{N}_0$ by Lemma 5. Now let K be any minimal set in Y and consider the restrictions of the linear skew-product flows π, π_{T} and π_{N} to the vector bundles $\mathscr{E}\,|_K$, $\mathscr{H}\,|_K$ and $\mathscr{N}\,|_K$ which are formed by restricting $\mathscr{E}$, $\mathscr{H}$ and $\mathscr{N}$ to the new base space K. One then has $\mathscr{B}_{\mathrm{N}}\,|_{K,\lambda} = (\mathscr{N}\,|_K)_0$ and by Theorem B(2), $\lambda \in \rho(\mathscr{N}\,|_K, \pi_{\mathrm{N}})$. It then follows from Theorem 1 that

$$\dim \mathscr{S}_{\mathrm{N},\lambda}(y) = \dim \mathscr{S}_\lambda(y) - \dim \mathscr{S}_{\mathrm{T},\lambda}(y) \tag{5.11}$$

for all $y \in K$, i.e., $\dim \mathscr{S}_{\mathrm{N},\lambda}(y)$ is constant over K. Now this argument applies to every minimal set $K \subseteq Y$. Since the right side of (5.11) is constant for $y \in Y$, we see that $\dim \mathscr{S}_{\mathrm{N},\lambda}(y)$ is constant for $y \in M$, where M is the union of all minimal sets in Y. Consequently by Theorem B(2) one has $\lambda \in \rho(\mathscr{N}, \pi_{\mathrm{N}})$. A similar argument applies when we assume that $\lambda \in \rho(\mathscr{N}, \pi_{\mathrm{N}}) \cap \rho(\mathscr{E}, \pi)$ and we conclude that (5.10) is valid. Q.E.D.

The following result is an immediate corollary of the last lemma.

Lemma 7. *Let I be a spectral interval in $\Sigma(\mathscr{H}, \pi_{\mathrm{T}})$ (or $\Sigma(\mathscr{N}, \pi_{\mathrm{N}})$). Assume that I is disjoint from $\Sigma(\mathscr{N}, \pi_{\mathrm{N}})$ (or $\Sigma(\mathscr{H}, \pi_{\mathrm{T}})$). Then I is a spectral interval in $\Sigma(\mathscr{E}, \pi)$.*

Remark. In the case described in Lemma 7 one can also show that the two invariant spectral bundles corresponding to the interval I have the same dimension.

We now have the following result.

THEOREM 2. *Assume that the Standing Hypothesis is satisfied. Then a sufficient condition for*

$$\Sigma(\mathscr{E}, \pi) = \Sigma(\mathscr{H}, \pi_T) \cup \Sigma(\mathscr{N}, \pi_N) \tag{5.12}$$

is either of the following three conditions:

(A) *The flow σ is chain recurrent on Y.*

(B) $\Sigma(\mathscr{H}, \pi_T) \cap \Sigma(\mathscr{N}, \pi_N) = \varnothing$.

(C) *$\mathscr{H}$ has an invariant complementary subbundle $\mathscr{M}$.*

Proof. Because of Theorem 1 it will suffice to show that

$$\lambda \in \rho(\mathscr{E}, \pi) \Rightarrow \lambda \in \rho(\mathscr{H}, \pi_T) \cap \rho(\mathscr{N}, \pi_N).$$

Let $\lambda \in \rho(\mathscr{E}, \pi)$. Then by Lemma 5 one has

$$\mathscr{B}_{T,\lambda} = \mathscr{H}_0 \qquad \text{and} \qquad \mathscr{B}_{N,\lambda} = \mathscr{N}_0 . \tag{5.13}$$

(A) If the flow σ on Y is chain recurrent, then it follows from (5.13) and Theorem B(3) that $\lambda \in \rho(\mathscr{H}, \pi_T) \cap \rho(\mathscr{N}, \pi_N)$.

(B) If $\Sigma(\mathscr{H}, \pi_T) \cap \Sigma(\mathscr{N}, \pi_N) = \varnothing$, then it follows from Lemma 7 that $\Sigma(\mathscr{H}, \pi_T) \subseteq \Sigma(\mathscr{E}, \pi)$ and $\Sigma(\mathscr{N}, \pi_N) \subseteq \Sigma(\mathscr{E}, \pi)$. Hence (5.12) is valid.

(C) Now assume that $\mathscr{H}$ has an invariant complementary bundle $\mathscr{M}$. Because of Lemma 4, there is no loss in generality in assuming that it is this bundle $\mathscr{N}$ that satisfies the Standing Hypothesis. Then the flow π_N on $\mathscr{N}$ is the restriction of π to $\mathscr{N}$. Let $\lambda \in \rho(\mathscr{E}, \pi)$. One then has following Whitney sum representation $\mathscr{E} = \mathscr{S}_\lambda + \mathscr{U}_\lambda = \mathscr{H} + \mathscr{N}$. For each $y \in Y$ we can find linear subspaces $\mathscr{K}_T(y)$ and $\mathscr{K}_N(y)$ in $\mathscr{H}(y)$ and $\mathscr{N}(y)$, respectively, so that

$$\mathscr{H}(y) = \mathscr{S}_{T,\lambda}(y) \oplus \mathscr{U}_{T,\lambda}(y) \oplus \mathscr{K}_T(y)$$
$$\mathscr{N}(y) = \mathscr{S}_{N,\lambda}(y) \oplus \mathscr{U}_{N,\lambda}(y) \oplus \mathscr{K}_N(y).$$

Since $\mathscr{S}_\lambda = \mathscr{S}_\lambda \cap \mathscr{E}$ and since $\mathscr{H}$ and $\mathscr{N}$ are invariant, it follows from (5.13) and [25, Part II, Lemma 6] that

$$\begin{aligned}\mathscr{S}_\lambda(y) = \mathscr{S}_\lambda(y) \cap (\mathscr{S}_{T,\lambda}(y) + \mathscr{S}_{N,\lambda}(y)) + \mathscr{S}_\lambda(y)\\ \cap (\mathscr{U}_{T,\lambda}(y) + \mathscr{K}_T(y) + \mathscr{U}_{N,\lambda}(y + \mathscr{K}_N(y))\\ = \mathscr{S}_\lambda(y) \cap (\mathscr{S}_{T,\lambda}(y) + \mathscr{S}_{N,\lambda}(y)) + 0 = \mathscr{S}_{T,\lambda}(y) + \mathscr{S}_{N,\lambda}(y).\end{aligned}$$

Similarly one has $\mathscr{U}_\lambda(y) = \mathscr{U}_{T,\lambda}(y) + \mathscr{U}_{N,\lambda}(y)$. It follows then that $\dim(\mathscr{K}_T(y) + \mathscr{K}_N(y)) = 0$. Since this holds for each $y \in Y$ one has $\lambda \in \rho(\mathscr{H}, \pi_T) \cap \rho(\mathscr{N}, \pi_N)$. Q.E.D.

The condition of disjointness

$$\Sigma(\mathscr{H}, \pi_T) \cap \Sigma(\mathscr{N}, \pi_N) = \varnothing$$

of the tangential and normal spectra occurs elsewhere in the literature. We shall discuss the history of this concept later. At this point we shall derive an interesting consequence of the disjointness condition.

THEOREM 3. *Assume that the Standing Hypothesis is satisfied and that one has*

$$\Sigma(\mathscr{H}, \pi_T) \cap \Sigma(\mathscr{N}, \pi_N) = \varnothing.$$

Then there exists an invariant complementary subbundle $\mathscr{M}$ for $\mathscr{H}$.

Proof. This fact is a simple consequence of the Spectral Theorem A and Theorem 2. Let $[a_i, b_i]$, $1 \leqslant i \leqslant k$, be an enumeration of the spectral intervals in $\Sigma(\mathscr{E}, \pi)$. Define the sets

$$A = [i: [a_i, b_i] \subseteq \Sigma(\mathscr{H}, \pi_T)\},$$
$$B = \{i: [a_i, b_i] \subseteq \Sigma(\mathscr{N}, \pi_N)\}.$$

Since $\Sigma(\mathscr{E}, \pi) = \Sigma(\mathscr{H}, \pi_T) \cup \Sigma(\mathscr{N}, \pi_N)$ and $\Sigma(\mathscr{H}, \pi_T) \cap \Sigma(\mathscr{N}, \pi_N) = \varnothing$, it follows that $\{A, B\}$ forms a partition of the set $\{1,..., k\}$. Now let $\mathscr{V}_i$ denote the spectral subbundle of $\mathscr{E}$ associated with the spectral interval $[a_i, b_i]$. We now leave it as an exercise to verify first that $\mathscr{H} = \Sigma_{i\in B} \mathscr{V}_i$, and second that $\mathscr{M} = \Sigma_{i\in A} \mathscr{V}_i$ satisfies the conclusions of the theorem. Q.E.D.

Remark. As noted in Section III, when $\mathscr{H}$ has an invariant complementary bundle $\mathscr{M}$, then π is a block diagonal system with respect to local coordinates $u \in \mathscr{H}(y)$ and $v \in \mathscr{M}(y)$.

VI. SPECTRA OF AN INVARIANT SUBMANIFOLD

Let M be a smooth n-dimensional manifold without boundary. (The manifold M need not be compact.) Let TM denote the tangent bundle to M. Let V denote a smooth vector field on M and let σ denote the local flow on M generated by V. This means that $\sigma(y, t)$ is the unique noncontinuable solution of

$$y' = V(y), \qquad y(0) = y$$

on M. If M is compact, then σ is a global flow, i.e., every motion $\sigma(y, t)$ is defined for all $t \in R$.

Let Y be a smooth, compact, connected submanifold of M of dimension k and assume that Y is an invariant set of the flow σ. This means that for every $y \in Y$ the vector $V(y)$ is tangent to Y at the point y, i.e., $V(y) \in T_yM$. Let $\mathscr{E}$ denote the subset of the tangent bundle TM consisting of

$$\mathscr{E} = \bigcup_{y \in Y} T_yM.$$

Then $\mathscr{E}$ is a vector bundle over Y and $\dim \mathscr{E}(y) = n = \dim M$ for all $y \in Y$. Furthermore $\mathscr{E}$ is an invariant set in the induced linearized flow on TM. We shall denote the induced linearized flow restricted to $\mathscr{E}$ by π. Thus in our usual convention we have

$$\pi(x, y, t) = (\varphi(x, y, t), y \cdot t),$$

where $\sigma(y, t) = y \cdot t$ and φ are solutions of the initial value problems

$$\begin{aligned} y' &= V(y), & y(0) &= y & (y &\in Y), \\ x' &= DV(y \cdot t)x, & x(0) &= x & (x &\in \mathscr{E}(y)) \end{aligned}$$

and $DV(y \cdot t)$ denotes the linear part of V evaluated along the trajectory $y \cdot t$.

We let $\mathscr{H}$ denote the tangent bundle TY. Then $\mathscr{H}$ is a vector bundle over Y with $\dim \mathscr{H}(y) = k = \dim Y$ for all $y \in Y$. Furthermore $\mathscr{H}$ is an invariant set for the linear skew-product flow π, i.e., $\mathscr{H}$ is an invariant subbundle in $\mathscr{E}$. Next let $\mathscr{N}$ be a complementary subbundle to $\mathscr{H}$. For example, if $\mathscr{E}$ has an inner product[1] $\langle \cdot, \cdot \rangle_y$ compatible with the topology on $\mathscr{E}$ then one could choose $\mathscr{N}$ to be the orthogonal complement $\mathscr{N} = \mathscr{H}^{\perp}$. The theory of Section V is now applicable to this situation. The three spectra

$$\Sigma(\mathscr{E}, \pi), \qquad \Sigma(\mathscr{H}, \pi_{\mathrm{T}}), \qquad \Sigma(\mathscr{N}, \pi_{\mathrm{N}})$$

are called, respectively, the *spectrum*, the *tangential spectrum* and the *normal spectrum* of Y. The tangential spectrum $\Sigma(\mathscr{H}, \pi_{\mathrm{T}})$ describes the exponential growth rates between pairs of solutions in Y itself. The normal spectrum $\Sigma(\mathscr{E}, \pi)$ describes the growth rates in the vicinity of Y. The various spectral bundles $\mathscr{V}_i$, which are described in Theorem A, give the precise location of the initial data for which the corresponding solution of the linearized equation has exponential growth rate in the ith-interval; cf. [26].

Let us now look at an illustrative example of a flow on a four-dimensional manifold M^4, which arises from a suitable Anosov flow.

[1] Such an inner product could be generated by a given Riemannian structure on M; cf. [11].

EXAMPLE. We begin with the Anosov diffeomorphism on T^2, the two-dimensional torus generated by the mapping $y = Ax$, where

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (\text{mod } 1). \tag{6.1}$$

The eigenvalues for this mapping are $\lambda = (3 \pm 5^{1/2})/2$ and the associated eigenvectors are

$$\lambda_1 = \frac{3}{2} + \frac{5^{1/2}}{2} \sim \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{5^{1/2}}{2} \end{pmatrix} = e_1 ,$$

$$\lambda_2 = \frac{3}{2} - \frac{5^{1/2}}{2} \sim \begin{pmatrix} 1 \\ \frac{1}{2} - \frac{5^{1/2}}{2} \end{pmatrix} = e_2 .$$

As is well known [18], the periodic points of A are dense in T^2. Over each point in T^2 the unstable manifold is Span(e_1) and the stable manifold is Span(e_2). The spectrum of A, as we defined it above, is formed by finding the solution σ_1 and σ_2 of the equations $e^{\sigma_i} = \lambda_i$, $i = 1, 2$, which is

$$\sigma_1 \doteq 0.962, \qquad \sigma_2 \doteq -0.962.$$

(Since $\lambda_1\lambda_2 = 1$ one has $\sigma_1 + \sigma_2 = 0$.)

By taking the suspension of the mapping $y = Ax$, one gets an Anosov flow on a 3-manifold M^3. Recall that M^3 can be identified as the collection of all $z = (x, s)$ where $x = (x_1 , x_2) \in R^2$, $s \in R^1$ and where $(x, s) \equiv (\hat{x}, \hat{s})$ if and only if $\hat{x} = A^n x$ and $\hat{s} = s + n$ for some integer n. The differential equation on M^3 for the Anosov flow has the form $z' = F(z)$ or

$$x_1' = f_1(x_1 , x_2 , s),\ x_2' = f_2(x_1 , x_2 , s) \qquad s' = 1. \tag{6.2}$$

The spectrum of this flow on M^3 is the set $\{\sigma_1 , -\sigma_1 , 0\}$.

Now consider the four-dimensional manifold $M^4 = M^3 \times S^1$. Introduce the coordinates $w = (z, \theta)$, where $z \in M^3$ and $\theta \in S^1$. Consider the differential system on M^4 given by

$$z' = F(z) + h(z, \theta, u),$$
$$\theta' = \alpha \sin \theta + g(z, \theta, \mu),$$

where α and μ are real parameters, and h and g are smooth functions that satisfy

$$h(z, 0, 0) = 0, \qquad | g(z, \theta, \mu)| = O(| \theta |^2).$$

Then at $\mu = 0$, there is an invariant manifold $Y = M^3$ given by $\theta = 0$. The induced flow on this manifold is the Anosov flow and consequently the tangential spectrum is

$$\Sigma_T = \{\sigma_1, -\sigma_1, 0\}$$

and the normal spectrum is $\Sigma_N = \{\alpha\}$. Thus if $\alpha \notin \Sigma_T$ one has $\Sigma = \{\alpha, \sigma_1, -\sigma_1, 0\}$.

If we require further that $|\alpha| > \sigma_1$ then we can apply [4, 14] to conclude that for all μ sufficiently small there exists a family of smooth manifolds Y_μ varying continuously in μ with $Y_0 = Y$. From the Spectral Perturbation Theorem [26, p. 342] there exist three disjoint compact spectral intervals I_μ^+, I_μ^0, I_μ^- such that the tangential spectrum $\Sigma_{T,\mu}$ of Y_μ satisfies $\Sigma_{T,\mu} = I_\mu^- \cup I_\mu^0 \cup I_\mu^-$, $0 \in I_\mu^0$, $I_\mu^- \subsetneq \{t \in R: t < 0\}$ and $I_\mu^+ \subsetneq \{t \in R: t > 0\}$. It follows that the induced flow on Y_μ is an Anosov flow for μ sufficiently small.

Remark. The argument used in the above example generalizes. If Y is a compact invariant submanifold of a flow on a manifold M and the tangential spectrum has the form

$$\Sigma = I^- \cup I^0 \cup I^+,$$

where I^-, I^0 and I^+ are disjoint compact sets with $0 \in I^0$, I^0 = an interval, $I^- \subsetneq \{t \in R: t < 0\}$, $I^+ \subset \{t \in R: t > 0\}$ and the dimension of the invariant spectral subbundle associated with I^0 is 1, then the flow on Y is an Anosov flow. The Perturbation Theorem [26, p. 432] insures that a small perturbation of the flow on Y is also an Anosov flow, which is known [3, 31].

VII. Block Triangular Differential Systems

The theory of Section V applies to another important class of problems, namely, linear differential equations with block triangular coefficient matrices. In this case one starts with a linear equation $x' = M(t)x$, where $M(t)$ is a block triangular matrix

$$M(t) = \begin{pmatrix} A(t) & B(t) \\ 0 & D(t) \end{pmatrix}$$

and A, B and D are respectively $(m \times m)$, $(n \times m)$ and $(n \times n)$ matrices for each $t \in R$. The vector x belongs to R^{m+n}.

As we have seen elsewhere [25], the linear equation $x' = M(t)x$ generates a linear skew-product flow π on a space $X \times \mathscr{F}$, where $\mathscr{F}$ is a suitable space of matrix valued functions and $M \in \mathscr{F}$. More precisely we assume that $\mathscr{F}$ is a compact, translation-invariant set. The invariance means that if $M \in \mathscr{F}$ and

$\tau \in R$, then $M_\tau \in \mathscr{F}$ where $M_\tau(t) = M(\tau + t)$. For every $M \in \mathscr{F}$ and $x \in X$ we let $\varphi(x, M, t)$ denote the solution of the initial value problem

$$x' = M(t)x, \qquad x(0) = x.$$

We assume further that the given topology on $\mathscr{F}$ is such that the mapping

$$\pi(x, M, \tau) = (\varphi(x, M, \tau), M_\tau) \tag{7.1}$$

is continuous. In this case π is the desired linear skew-product flow. See [25] for more details.

If we assume, as we do, that M is block triangular, then there is no loss in generality in assuming that $\mathscr{F}$ consists entirely of block triangular matrices of the same form.

The differential system $x' = M(t)x$ becomes

$$\begin{aligned} u' &= A(t)u - B(t)v, \\ v' &= \qquad\qquad D(t)v, \end{aligned} \tag{7.2}$$

where $x = \binom{u}{v}$, $u \in R^m$ and $v \in R^n$. It is easily seen that the fundamental solution $\Phi(M, t)$ has the same block triangular form, and thus the linear skew-product flow (7.1) is a "block triangular system" in the sense defined in Section III. The invariant subbundle $\mathscr{H}$ is

$$\mathscr{H} = (R^m \times \{0\}) \times \mathscr{F}.$$

An initial vector $x = \binom{u}{v}$ lies in $\mathscr{H}(M)$, $M \in \mathscr{F}$, if and if $v = 0$. The flow π_{T} on $\mathscr{H}$ is simply

$$\pi_{\mathrm{T}}(u, M, \tau) = (\Phi(A, \tau)u, M_\tau),$$

where $\Phi(A, t)$ is the fundamental solution of $u' = A(t)u$, and $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$.

The associated normal subbundle is

$$\mathscr{N} = (\{0\} \times R^n) \times \mathscr{F}.$$

An initial vector $x = \binom{u}{v}$ lies in $\mathscr{N}(M)$, $M \in \mathscr{F}$, if and only if $u = 0$. The induced normal flow π_{N} on $\mathscr{N}$ is given by

$$\pi_{\mathrm{N}}(v, M, \tau) = (\Phi(D, \tau)v, M_\tau),$$

where $\Phi(D, t)$ is the fundamental solution of $v' = D(t)v$ and M is given above.

The linear skew-product flow π given by (7.1) is "smooth" and the variation of constants formula (3.5) takes on the classical form

$$\begin{aligned} u_t &= \Phi_\lambda(A, t) \left[u + \int_0^t \Phi_\lambda^{-1}(A, s)\, B(s)\, \Phi_\lambda(D, s) v\, ds\right], \\ v_t &= \Phi_\lambda(D, t) v, \end{aligned} \tag{7.3}$$

where $\Phi_\lambda(A, t) = e^{-\lambda t}\Phi(A, t)$ and $\Phi_\lambda(D, t) = e^{-\lambda t}\Phi(D, t)$.

Theorems 1, 2 and 3 are now applicable to the system (7.2). In particular one has $\Sigma \subseteq \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{N}}$ where Σ, Σ_{T} and Σ_{N} denote the spectra of π, π_{T} and π_{N}, respectively. In the next example we show that the equality

$$\Sigma = \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{N}}$$

can fail if none of the sufficient conditions of Theorem 2 are satisfied.

EXAMPLE. For τ, $t \in R$ define the (2×2) matrix-valued function

$$M_\tau(t) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, & t > -\tau, \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, & t < -\tau. \end{cases}$$

In the space $L_{1,\mathrm{loc}}$ the hull $\mathscr{F} = H(M_0) = \mathrm{Cl}\{M_\tau: \tau \in R\}$ is compact. It is this space $\mathscr{F}$, with the $L_{1,\mathrm{loc}}$-topology, which is the base space of our linear skew-product flow. It is not difficult to see that

$$\mathscr{F} = A(M_0) \cup \Omega(M_0) \cup \{M_\tau: \tau \in R\},$$

where $A(M_0)$ and $\Omega(M_0)$ are the alpha and omega limit sets of M_0. Furthermore $A(M_0)$ and $\Omega(M_0)$ consist of two constant coefficient matrices, viz.,

$$M_- = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_+ = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

For $M \in \mathscr{F}$ and $x \in R^2$ we let

$$\pi(x, M, \tau) = (\varphi(x, M, \tau), M_\tau)$$

denote the motion through (x, M). Let Σ denote the spectrum of the flow π.

Let A, D and B be defined by $M_0 = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$, then

$$A(t) = \begin{cases} 1, & t > 0, \\ -1, & t < 0, \end{cases} \qquad D(t) = \begin{cases} -1, & t > 0, \\ 1, & t < 0, \end{cases}$$

and $B(t) \equiv 1$. Furthermore one has

$$\Phi_\lambda(A, t) = \begin{cases} e^{(1-\lambda)t}, & t > 0, \\ e^{(-1-\lambda)t}, & t < 0, \end{cases} \qquad \Phi_\lambda(D, t) = \begin{cases} e^{(-1-\lambda)t}, & t > 0, \\ e^{(1-\lambda)t}, & t < 0, \end{cases}$$

where $\lambda \in R$. Therefore it follows from (7.3) that

$$\begin{aligned} u_t &= e^{(1-\lambda)t}\left\{u + \frac{v}{2}\right\} - e^{(-1-\lambda)t}\frac{v}{2}, \\ v_t &= e^{(-1-\lambda)t}v \end{aligned} \tag{7.4}$$

for $t > 0$, and for $t < 0$ one has

$$\begin{aligned} u_t &= e^{(-1-\lambda)t}\left\{u - \frac{v}{2}\right\} + e^{(1-\lambda)t}\frac{v}{2}, \\ v_t &= e^{(1-\lambda)t}v. \end{aligned} \tag{7.5}$$

FACT 1. $\Sigma = \{-1\} \cup \{1\}$.

In order to prove this we first fix λ satisfying $-1 < \lambda < 1$. In this case it follows from (7.4) that u_t and v_t are bounded for $t \geqslant 0$ if and only if $u + v/2 = 0$. From (7.5) it follows that u_t and v_t are bounded for $t \leqslant 0$ if and only if $u - v/2 = 0$. Consequently the bounded set $\mathscr{B}_\lambda(M_0)$ is trivial, i.e., $\mathscr{B}_\lambda(M_0) = \{0\}$. It follows that for every translate M_τ one has $B_\lambda(M_\tau) = \{0\}$. Also one has $\mathscr{B}_\lambda(M_-) = \mathscr{B}_\lambda(M_+) = \{0\}$. Thus one has $\mathscr{B}_\lambda = \{0\} \times \mathscr{F}$. Since the exponential dichotomies over $A(M_0)$ and $\Omega(M_0)$ are compatible, it follows from Theorem B(2) that λ is in the resolvent ρ. In a similar way one can show that if $\lambda < -1$, or $\lambda > 1$, then $\lambda \in \rho$. Therefore one has $\Sigma \subseteq \{-1\} \cup \{1\}$. Finally if $\lambda = \pm 1$ one can see that $\mathscr{B}_\lambda \neq \{0\} \times \mathscr{F}$, and consequently it follows that $\Sigma = \{-1\} \cup \{1\}$.

The tangential flow π_T is given by $\pi_T(u, M, \tau) = (\Phi(A, \tau)u, M_\tau)$ and the normal flow π_N by $\pi_N(v, M, \tau) = (\Phi(D, \tau)u, M_\tau)$. The following fact is easily verified.

FACT 2. $\Sigma_T = \Sigma_N = [-1, 1]$.

Hence one has $\Sigma \neq \Sigma_T \cup \Sigma_N$.

Remark 1. The last example leads to other consequences. Specifically it is shown in [24] that there is a connection between the above spectral theory, and the Massera–Schäffer theory of admissibility and the classical theory of the Fredholm index.

2. In the last example we see that the boundary of Σ_T, $\partial\Sigma_T$, lies in Σ. It is not difficult to show in general that one has $\partial\Sigma_T \subseteq \Sigma$. Also one can show that if I is any component of $\Sigma_T \cup \Sigma_N$ then $\Sigma \cap I \neq \varnothing$.

Note added in proof. After this manuscript was completed we became aware of the following related papers: (1) C. CHICONE AND R. SWANSON, *J. Differential Equations* **36** (1980), 28–40. (2) C. CHICONE AND R. SWANSON, Spectral theory for linearizations of dynamical systems, *J. Differential Equations* (in press). (3) R. SWANSON, The spectrum of vector bundle flows with invariant subbundles (preprint).

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